

EQUALITY CASES IN VITERBO'S CONJECTURE RELATED TO PERMUTOHEDRA

ALEXEY BALITSKIY

ABSTRACT. In this note we show, using the billiard technique, that the product of a regular permutohedron and a regular simplex delivers an equality in Viterbo's conjecture.

1. INTRODUCTION

C. Viterbo conjectured [8] an isoperimetric inequality for a certain symplectic capacity and the volume of a convex body $X \subset \mathbb{R}^{2n}$:

$$\text{vol}(X) \geq \frac{c(X)^n}{n!}.$$

The minimum is supposed to be attained on balls. In this note we study certain other convex bodies for which the equality also holds, for which we do not know if they are balls or not. The bodies we consider are Lagrangian products of something in $V = \mathbb{R}^n$ with q coordinates and something in $V^* = \mathbb{R}^n$ with p coordinates:

$$\text{vol}(P_n \times \triangle_n^\circ) = \frac{c_{HZ}(P_n \times \triangle_n^\circ)^n}{n!},$$

where \triangle_n denotes a regular n -dimensional simplex, and P_n denotes the n -dimensional permutohedron, which can be considered as the Minkowski sum of all the edges of \triangle_n (this definition can be found in [9, Lecture 7.3]).

The connection between symplectic capacities of Lagrangian products and billiards, established in [3], can be stated as

$$\xi_T(K) = c_{HZ}(K \times T),$$

where $c_{HZ}(\cdot)$ stands for the Hofer–Zehnder symplectic capacity and $\xi_T(K)$ denotes the length of the shortest closed billiard trajectory in a convex body $K \subset V$ with geometry of lengths given by another convex body $T \subset V^*$ and its norm

$$\|q\|_T = \max_{p \in T} \langle p, q \rangle$$

in V . In Section 2 we introduce the billiard definition in detail.

The shortest closed billiard trajectory is actually understood in the sense of K. Bezdek and D. Bezdek [5], as the closed polygonal line of minimal $\|\cdot\|_T$ length not fitting into a translate of the interior of K .

By a *classical billiard trajectory* we mean a trajectory that, considered as a polygonal line in $\partial(K \times T)$, only meets smooth points of ∂K and smooth points of ∂T . We also do not allow a classical trajectory to pass the same path multiply.

In [4] we considered some particular cases, which we summarize in following statements:

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Proposition 1.1. (1) *In the configuration $(\Delta_2 - \Delta_2) \times \Delta_2^\circ$ (our regular simplices are always centered at the origin) the shortest generalized billiard trajectory has length*

$$\xi_{\Delta_2^\circ}(\Delta_2 - \Delta_2) = 9,$$

so $X = (\Delta_2 - \Delta_2) \times \Delta_2^\circ$ delivers equality in Viterbo's conjecture.

(2) *Any simple classical billiard trajectory in the configuration*

$$(\Delta_2 - \Delta_2) \times \Delta_2^\circ$$

bounces 4 times and has length 9.

(3) *Also, arbitrarily close to any point*

$$(p, q) \in \partial((\Delta_2 - \Delta_2) \times \Delta_2^\circ)$$

there passes a certain classical billiard trajectory of minimal length.

In the same paper the following peculiar property of Hanner's polytopes (which are famous for delivering the equality to Mahler's conjecture) was established:

Proposition 1.2. (1) *In a Hanner polytope $H \subset \mathbb{R}^n$ with geometry specified by its polar H° the shortest generalized billiard trajectory has length $\xi_{H^\circ}(H) = 4$ and $X = H \times H^\circ$ delivers equality in Viterbo's conjecture (the latter fact was known since [2]).*

(2) *Any simple classical billiard trajectory in a Hanner polytope $H \subset \mathbb{R}^n$ with geometry specified by its polar H° is $2n$ -periodic and has length 4.*

(3) *Moreover, in an arbitrarily small neighborhood of any point $(q, p) \in \partial(H \times H^\circ)$ there passes a classical billiard (in configuration $H \times H^\circ$) trajectory of minimal length.*

In Section 3 we establish partial similar result for the configuration $P_n \times \Delta_n^\circ$, which turns out to be the direct generalization of the configuration $(\Delta_2 - \Delta_2) \times \Delta_2^\circ$ in the plane.

F. Schlenk has established (see [7]) that the interior of the Lagrangian product of a crosspolytope and a cube (its polar) equals the interior of a certain ball from symplectic point of view. The central question is whether such a situation (when shortest billiards trajectories are dense on $\partial(K \times T)$) implies symplectomorphicity to a ball, or at least implies equality in Viterbo's conjecture.

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2. BILLIARDS IN MINKOWSKI NORM

Let an n -dimensional real vector space $V = \mathbb{R}^n$ be endowed with a norm with unit ball T° (where $T^\circ \subset V$ is polar to a convex body $T \subset V^*$). We follow the notation of [1] and denote such a norm by $\|\cdot\|_T$. By definition, $\|q\|_T = \max_{p \in T} \langle p, q \rangle$, where $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{R}$

is the canonical bilinear form of the duality between V and V^* . Here we assume that T contains the origin, but is not necessarily centrally symmetric. Therefore the norm may be non-symmetric, in general, $\|q\|_T \neq \|-q\|_T$ (sometimes, such norms are called *gauges*).

The *momentum* $p \in \partial T \subset V^*$ of the trajectory fragment $q \rightarrow q'$ is defined as a functional reaching its maximum at $q' - q$ (if T is not strictly convex then there is an ambiguity in the definition of p).

The cone $N_K(q)$ of outer normals is defined by

$$N_K(q) = \{n \in V^* : \langle n, q' - q \rangle \leq 0 \ \forall q' \in K\}.$$

The *generalized reflection law* is the following:

$$(2.1) \quad p' - p \in -N_K(q),$$

where p and p' stand for momenta of billiard trajectory before and after the reflection at the point q .

We put

$$\begin{aligned}\mathcal{P}_m(K) &= \{(q_1, \dots, q_m) : \{q_1, \dots, q_m\} \text{ doesn't fit into } (\text{int } K + t) \text{ with } t \in V\} = \\ &= \{(q_1, \dots, q_m) : \{q_1, \dots, q_m\} \text{ doesn't fit into } (\alpha K + t) \text{ with } \alpha \in (0, 1), t \in V\}\end{aligned}$$

and

$$\xi_T(K) = \min_{Q \in \mathcal{Q}_T(K)} \ell_T(Q),$$

where $Q = (q_1, \dots, q_m)$, $m \geq 2$, ranges over the set $\mathcal{Q}_T(K)$ of all closed generalized billiard trajectories in K with geometry defined by T . (Here we denote the length $\ell_T(q_1, \dots, q_m) = \sum_{i=1}^m \|q_{i+1} - q_i\|_{T \cdot}$)

The generalization of the main result of [5], proved in [1], states the following:

Theorem 2.1. *For any convex bodies $K \subset V, T \subset V^*$ (T is smooth) containing the origins of V and V^* in their interiors, the equality holds:*

$$\xi_T(K) = \min_{m \geq 2} \min_{Q \in \mathcal{P}_m(K)} \ell_T(Q);$$

and furthermore, the minimum is attained at $m \leq n + 1$.

Remark 2.2. Actually, Theorem 2.1 is proved in [1] for smooth body K and classical trajectories. But approximating non-smooth K in Hausdorff metrics and passing to the limit we obtain the formulation above. Note that the formula of Theorem 2.1 can be used as the definition of $\xi_T(K)$ for arbitrary T and K without any smoothness assumptions.

3. STATEMENT

Consider a regular simplex $\Delta_n = \text{conv}\{v_0, \dots, v_n\} \subset \mathbb{R}^n$, normalized so that its edges are all of unit length. Choose an orthonormal base (e_1, \dots, e_n) with e_n pointing to v_0 . Consider also the permutohedron $P_n \subset \mathbb{R}^n$, defining it as the Minkowski sum of simplex edges:

$$P_n = \sum_{0 \leq i < j \leq n} [v_i, v_j].$$

So the main result is:

Theorem 3.1.

$$\text{vol}(P_n \times \Delta_n^\circ) = \frac{c_{HZ}(P_n \times \Delta_n^\circ)^n}{n!}.$$

Evidently, the equality follows from the following two propositions.

Proposition 3.2. *In above notation*

$$\text{vol } \Delta_n^\circ = \frac{2^{n/2}(n+1)^{n+1/2}}{n!}, \quad \text{vol } P_n = \frac{(n+1)^{n-1/2}}{2^{n/2}}.$$

Proposition 3.3. *In above notation*

$$c_{HZ}(P_n \times \Delta_n^\circ) = (n+1)^2.$$

4. PROOFS

Proof of Proposition 3.2. The volume $\text{vol } \Delta_n = \frac{\sqrt{n+1}}{2^{n/2}n!}$ can be easily computed, and the Mahler volume product $\text{vol } \Delta_n \cdot \text{vol } \Delta_n^\circ = \frac{(n+1)^{n+1}}{(n!)^2}$ is well known, so we do not give details here.

To compute $\text{vol } P_n$ we use another definition of the permutohedron (see [6, Chapter 21]): \widetilde{P}_n is defined as the Voronoi cell of the lattice

$$A_n^* = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_i x_i = 0, x_0 \equiv \dots \equiv x_n \pmod{n+1}\}$$

generated by vectors

$$a_1 = (\underbrace{-1, n, -1, \dots, -1}_{n+1})^t, \dots, a_n = (-1, -1, -1, \dots, n)^t.$$

Note that \widetilde{P}_n is $\sqrt{2}$ times larger than P_n (since the width of P_n equals $n\langle v_0 - v_1, e_n \rangle = \sqrt{(n^2 + n)/2}$, and the width of \widetilde{P}_n equals $|a_1| = \sqrt{n^2 + n}$). Then

$$2^{n/2} \text{vol } P_n = \text{vol } \widetilde{P}_n = \det A_n^* = \det \Gamma(a_1, \dots, a_n)^{1/2} = \begin{vmatrix} n^2 + n & -n - 1 & -n - 1 & \vdots \\ -n - 1 & n^2 + n & -n - 1 & \vdots \\ -n - 1 & -n - 1 & n^2 + n & \vdots \\ \dots & \dots & \dots & \ddots \end{vmatrix}^{1/2}$$

The latter $n \times n$ matrix has an eigenvector $h = (1, \dots, 1)^t$ corresponding to the eigenvalue $n+1$ without multiplicities. All other eigenvectors, orthogonal to h , have the same $(n-1)$ -fold eigenvalue $(n+1)^2$, as it is easy to check by hand. Therefore, $\det \Gamma(a_1, \dots, a_n) = (n+1)^{2n-1}$ and the second result follows. \square

To prove Proposition 3.3 we use Bezdeks' characterization of shortest generalized billiard trajectory in P_n , when lengths are measured using the norm with unit body Δ_n . Note that 2-periodic trajectory (2-fold bypass of the width of P_n) delivers an example of the trajectory of Δ_n -length $(n+1)^2$, that cannot fit into $\text{int } P_n$.

Further, we consider an arbitrary closed polygonal line that cannot fit into $\text{int } P_n$ and show its length cannot be less than $(n+1)^2$.

First, we replace each segment $[q, q']$ of this line with certain polygonal line of the same length but edges directed along v_0, \dots, v_n . This can be done as follows: Consider the convex cone

$$C_i = \text{cone}\{\{v_0, \dots, v_n\} \setminus \{v_i\}\},$$

in which the vector $q' - q$ lies, and decompose $q' - q = \sum_{0 \leq k \leq n, k \neq i} \alpha_k v_k$ with $\alpha_k \geq 0$. Now the polygonal line with edges congruent to $\alpha_i v_i$ suits well for our purpose. Its length equals the length of $q' - q$ in the given norm, because the norm function is linear on the considered cone.

Now, we have the closed polygonal line with at most $n+1$ directions used. Note that $\sum_i v_i = 0$ is the only linear dependence between the directions, thus the total length of the segments of this polygonal line along each direction v_i does not depend on i . Assume the contrary to the statement of Proposition 3.3: This length along a direction v_i is less than $n+1$. We show that such a line can fit into smaller homothet of P_n . It remains to establish the following:

Lemma 4.1. *Suppose a closed polygonal line only uses segments directed along v_0, \dots, v_n , and the total length along each of these direction equals $n+1$. Then this line can be covered by a translate of P_n .*

Proof. We proceed by induction on n . Base $n = 1$ is clear.

To begin, we note that top horizontal (meaning orthogonal to e_n , which is supposed to be directed upward) facet F of P_n is congruent to P_{n-1} placed horizontally in \mathbb{R}^n . Explicitly,

$$F = nv_0 + \sum_{1 \leq i < j \leq n} [v_i, v_j].$$

Now let $Q = q_0 \rightarrow q_1 \rightarrow \dots \rightarrow q_{m-1} \rightarrow q_m = q_0$ be a polygonal line satisfying the assumptions of the lemma, and let q_0 be the highest (in the sense of largest coordinate along e_n) vertex of Q . Let us introduce a parameterization $q : [0, n+1] \rightarrow Q$ (without loss of generality $q(0) = q(n+1) = 0$), so that the point $q(t)$ runs along Q with constant velocity, and the portion of time when point travels along any v_i direction equals 1.

Consider the following transformation of Q : we contract it using the transform

$$\begin{pmatrix} \frac{1}{n+1} & 0 & \vdots & 0 & 0 \\ 0 & \frac{1}{n+1} & \vdots & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & \vdots & \frac{1}{n+1} & 0 \\ 0 & 0 & \vdots & 0 & 1 \end{pmatrix}$$

and obtain the line \tilde{Q} with corresponding parameterization $\tilde{q}(\cdot)$. Also denote by \tilde{v}_i the image of v_i under this transform (note that $(n+1)\tilde{v}_i = v_i - v_0$).

Now we consider the Minkowski sum

$$R = \tilde{Q} + F = \bigcup_{t \in [0, n+1]} (F + \tilde{q}(t))$$

and claim that Q can be covered by R . This can be seen as follows: we need $Q \subset R + s$ for some $s \in \mathbb{R}^n$. Imagine that we track the relative motion of the point $q(t)$ inside the moving horizontal permutohedron $F + s + \tilde{q}(t)$ in the same horizontal hyperplane. The relative velocity of $q(t)$ in $F + s + \tilde{q}(t)$ equals $\frac{n}{n+1}$ -fraction of the corresponding horizontal velocity component of $q(t)$. Therefore, for this relative motion the inductive assumption holds: In permutohedron P_{n-1} we have the trajectory that travels by the distance n along each of the n distinguished directions (defined as horizontal projections of v_1, \dots, v_n). Therefore such s indeed can be found.

Next step is to cover the set R by P_n . To do this, we consider the coordinates $(\tilde{q}_1(t), \dots, \tilde{q}_n(t))$ of the point $\tilde{q}(t)$ in the base $\tilde{v}_1, \dots, \tilde{v}_n$. Shift \tilde{Q} (and $R = \tilde{Q} + F$ correspondingly) so that for any i

$$\min_{t \in [0, n+1]} \tilde{q}_i(t) = 0.$$

We claim that $R \subset P_n$ after such a shift. Indeed, any point of R could be represented as

$$f + \tilde{q}(t) = f + \sum_{i=1}^n \tilde{q}_i(t) \tilde{v}_i,$$

for some $t \in [0, n+1]$ and $f \in F$. Note that $\tilde{q}_i(t) \in [0, n+1]$ and the point $\sum_{i=1}^n \tilde{q}_i(t) \tilde{v}_i$ belongs to the Minkowski sum of the non-horizontal segments

$$\sum_{i=1}^n [0, (n+1) \tilde{v}_i] = \sum_{i=1}^n ([v_0, v_i] - v_0).$$

Hence

$$f + \sum_{i=1}^n \tilde{q}_i(t) \tilde{v}_i \in F + \sum_{i=1}^n ([v_0, v_i] - v_0) = \sum_{1 \leq i < j \leq n} [v_i, v_j] + \sum_{i=1}^n [v_0, v_i] = P_n,$$

as required. \square

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E-mail address: alexey_m39@mail.ru

DEPT. OF MATHEMATICS, MOSCOW INSTITUTE OF PHYSICS AND TECHNOLOGY, INSTITUTSKIY PER. 9, DOLGOPRUDNY, RUSSIA 141700

INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS RAS, BOLSHOY KARETNY PER. 19, MOSCOW, RUSSIA 127994